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EQUATIONS OF ELECTRICAL MACHINES WITH A ROTATING PERMANENT MAGNET PLAYING THE ROLE OF THE STATOR AND THEIR NONLOCAL ANALYSIS

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A complete mathematical model is developed for the motion of a current loop powered from a constant voltage source and placed in the field of a permanent magnet rotating with a constant angular velocity. Local analysis of this model shows that it is unstable in the absence of external load, which contradicts the practice of motor operation. Therefore, the motor rotor model considered is incorrect although it is frequently used. The detected contradiction is eliminated by introducing an additional loop, which is orthogonal to the initial one and has the same parameters but is short-circuited. The complete mathematical model of such a system is unstable in the absence of external load. For the case of an induction motor, the conditions of dichotomy, global asymptotic stability, and instability are formulated.

We consider a model of an electrical machine (EM) in which the stator is a rotating permanent magnet and the rotor is a current loop. The magnetic field of the EM stator is represented by a magnetic induction vector $\boldsymbol{B}$ of constant modulus, which rotates in a plane perpendicular to the rotation axis of the EM rotor with a constant angular velocity $\omega = 314$ sec^{-1}, equal to the angular frequency of the circuit voltage ($\omega = 2\pi f$ and $f = 50$ Hz). We first consider a rotor model in the form of an electric loop, to which a constant excitation voltage $u_f \geqslant 0$ is applied through the collector.

It should be noted that the condition $u_f > 0$ corresponds to the case of a synchronous motor, and $u_f = 0$ corresponds to the case of an induction motor. This agrees with physical concepts because a synchronous motor is an induction motor supplemented by an excitation coil. The latter produces an additional generalized force, which shifts the equilibrium position of the induction motor and allows the synchronous motor to operate in a synchronous mode under load.

The goal of the present work is to derive a complete system of equations and to analyze its stability.

In formulating the equations of motion of the indicated electromechanical system (Fig. 1), we use the Lagrange–Maxwell equations. The kinetic energy T and the dissipative function D are given by

$$T = T_{\rm m} + T_{\rm e} + T_{\rm e.m.}, \qquad D = D_{\rm m} + D_{\rm e},$$

where $T_{\rm m}$ is the mechanokinetic energy, $T_{\rm e}$ electrokinetic energy, $T_{\rm e.m.}$ is the electromechanokinetic energy, and $D_{\rm m}$ and $D_{\rm e}$ are the mechanical and electrical components of the dissipative function, respectively.

Apparently, Maxwell [1] was the first to represent the kinetic energy of an electromechanical system as a sum of three terms. He called the third term electroponderokinetic energy. To determine this component, which contains products of mass point velocities and values of electric current, Maxwell performed experiments, which shows that the contribution of electroponderokinetic energy to the total energy of the system is so inappreciable that it cannot be detected by measuring instruments. This conclusion was confirmed in the beginning of the XX century using more perfect measuring equipment. In what follows, we assume that $T_{\rm e.m.} = 0$.

It is known that in an EM, the total power of aerodynamic losses and bearing friction losses is a few tenths of percent of the rated power [2, p. 165]. Hence, the value of $D_{\rm m}$ is negligibly small and we can set $D_{\rm m} = 0$. It

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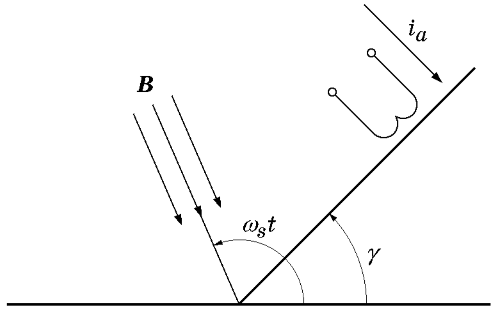

Fig. 1

should be noted that the last assumption is of no significance for further consideration. It is usually assumed that the bearing friction moment is constant and the moment generated by aerodynamic drag is proportional to the angular velocity. Therefore, such moments can be treated as components of the external load moment.

The energies $T_{\rm m}$ and $T_{\rm e}$ and the dissipative function $D_{\rm e}$ are written as

$$T_{\rm m} = J\dot{\gamma}^2/2, \qquad T_{\rm e} = L_a i_a^2/2 + \Psi i_a \cos(\omega t - \gamma), \qquad D_{\rm e} = R_a i_a^2/2,$$

where J is the moment of inertia of the current loop, γ is the angle of rotation of the magnetic axis of the loop passing through the center of the loop perpendicular to its plane, L_a is the inductance of the current loop, i_a is the loop current, $\Psi = BSw = \text{const}$ is the amplitude of interlinkage of the external magnetic field with the loop, $B = \text{const}$ is the modulus of the external magnetic induction vector, S is the area of the current loop, w is the number of sections in the loop, and R_a is the loop resistance.

The Lagrange–Maxwell equation for the independent electrical variable i_a is written as

$$\frac{d}{dt}\frac{\partial T_{\rm e}}{\partial i_a} + \frac{\partial D_{\rm e}}{\partial i_a} = u_f$$

or

$$L_a \dot{i}_a - \Psi(\omega - \dot{\gamma})\sin(\omega t - \gamma) + R_a i_a = u_f, \tag{1}$$

where u_f is the constant excitation voltage applied to the current loop and generalized over the coordinate i_a.

The Lagrange–Maxwell equation for the independent geometrical coordinate γ has the form

$$\frac{d}{dt}\frac{\partial T_{\rm m}}{\partial \dot{\gamma}} - \frac{\partial T_{\rm e}}{\partial \gamma} + M_{\rm load} = 0$$

($M_{\rm load}$ is the external load moment applied to the current loop) or

$$J\ddot{\gamma} = \Psi i_a \sin(\omega t - \gamma) - M_{\rm load}. \tag{2}$$

System (1), (2) completely describes the dynamics of the examined electromechanical system.
Let us introduce the load angle

$$\theta = \omega t - \gamma \tag{3}$$

and the slide s of the current loop relative to the external magnetic field

$$s = (\omega - \dot{\gamma})/\omega. \tag{4}$$

From (3) and (4) it follows that $\dot{\theta} = \omega s$.

Instead of the current time t, we introduce the dimensionless synchronous time $\tau = \omega t$, which corresponds to the angle of rotation of the external magnetic field (vector $\boldsymbol{B}$). Then, Eqs. (1) and (2) become

$$L_a\omega\frac{di_a}{d\tau} - \omega\Psi\Big(1 - \frac{d\gamma}{d\tau}\Big)\sin\theta + R_a i_a = u_f, \qquad \frac{d\theta}{d\tau} = s = 1 - \frac{d\gamma}{d\tau}, \qquad \frac{ds}{d\tau} = -\frac{1}{\omega^2 J}(\Psi i_a \sin\theta - M_{\rm load}). \tag{5}$$

To study system (5), it is reasonable to write it in dimensionless form by introducing the dimensionless current $\bar{i}_a$, the external load moment $\bar{M}_{\rm load}$, the voltage $\bar{u}_f$, and the interlinkage $\bar{\Psi}$:

$$i_a = \frac{u}{\omega L_a}\,\bar{i}_a, \quad M_{\rm load} = \frac{u\Psi}{\omega L_a}\,\bar{M}_{\rm load}, \quad u_f = \frac{uR_a}{\omega L_a}\,\bar{u}_f, \quad \Psi = \frac{u}{\omega}\,\bar{\Psi}.$$

Here u is the basis voltage (for example, the peak value of the circuit voltage).

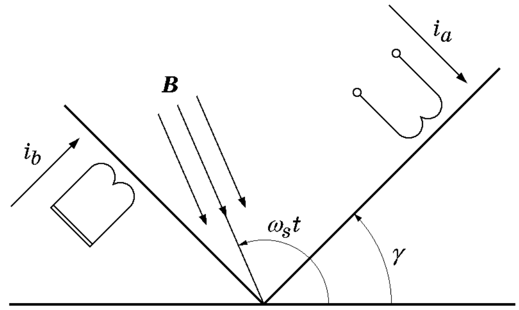

Fig. 2

In the dimensionless variables, system (5) takes the following form (the bar above the dimensionless quantities is dropped, and the point above the variable denotes differentiation with respect to the dimensionless time τ):

$$\dot{i}_a = -\alpha_\mathrm{r} i_a + bs\sin\theta + \alpha_\mathrm{r} u_f, \qquad \dot{\theta} = s, \qquad \dot{s} = -\delta(i_a \sin\theta - M_\mathrm{load}). \tag{6}$$

Here $\alpha_\mathrm{r} = R_a/(\omega L_a)$, $b = \bar{\Psi}$, and $\delta = u\Psi/(\omega^3 L_a J)$.

For the steady-state regime, from (6) we obtain $s = 0$, $\theta = \theta_0$, and $i_a = u_f$. The equilibrium condition for the electromagnetic moment and the load moment lead to the following equation for θ_0:

$$u_f \sin\theta_0 = M_\mathrm{load}. \tag{7}$$

This implies the existence of two equilibrium positions for $M_\mathrm{load} < u_f$.

Let us consider small oscillations of system (6) about the equilibrium position. The corresponding linear system is written as

$$\dot{\tilde{i}}_a = -\alpha_\mathrm{r}\tilde{i}_a + b\tilde{s}\sin\theta_0, \qquad \dot{\tilde{\theta}} = \tilde{s}, \qquad \dot{\tilde{s}} = -\delta(\tilde{i}_a \sin\theta_0 + u_f\tilde{\theta}\cos\theta_0), \tag{8}$$

where the values of θ_0 are determined from Eq. (7).

The characteristic equation of system (8) is as follows:

$$\lambda^3 + \alpha_\mathrm{r}\lambda^2 + \delta(b\sin^2\theta_0 + u_f\cos\theta_0)\lambda + \delta u_f\alpha_\mathrm{r}\cos\theta_0 = 0. \tag{9}$$

The Hurwitz minors for the polynomial in (9) satisfy the conditions

$$\Delta_1 = \alpha_\mathrm{r} > 0, \quad \Delta_2 = \delta\alpha_\mathrm{r} b\sin^2\theta_0 \geqslant 0, \quad \Delta_3 = \Delta_2\delta u_f\alpha_\mathrm{r}\cos\theta_0 \geqslant 0.$$

The second and third minors are equal to zero under no-load conditions, in which $M_\mathrm{load} = 0$. No-load conditions are natural operation conditions for almost all EM. None of them can operate in an unstable regime. This inconsistency of theory and practice is due to the choice of a one-winding (loop) model of the EM rotor.

It should be noted that equations equivalent to Eqs. (6) are obtained from Gorev's equations for a EM without damper loops [3] if the stator windings in them are considered superconducting and the magnetic field of the stator is considered steady-state. In [3], as in the case considered, the rotor is simulated by just one loop. In [4, 5] it is proved that in Gorev's equations [3, p. 192], the presence of resistances in the stator windings extends the zone of instability. This model is unstable not only under no-load conditions but also under small loads. The occurrence of the zone of instability is caused by nonzero resistance of the stator windings (see [5, p. 225]).

As noted above, one loop (winding) cannot simulate the cylindrical nature of the magnetic field in the air gap of an EM formed by the magnetic leads of the stator and rotor. A circular field can be simulated by at least two orthogonal loops (winding). To avoid the above discordance between theory and practice, the rotor loop (winding) should be supplemented by a loop (winding) similar to the one available but orthogonal to it and short-circuited.

We call such loop (winding) a phantom one and show that the indicated discordance disappears in its presence. For this, we repeat the calculations given above taking into account the phantom loop (Fig. 2).

The expressions for the kinetic energy and the dissipative function are written as

$$T = T_\mathrm{m} + T_\mathrm{e}, \qquad T_\mathrm{m} = J\dot{\gamma}^2/2,$$

$$T_\mathrm{e} = L_\mathrm{r}(i_a^2 + i_b^2)/2 + \Psi[i_a\cos(\omega t - \gamma) - i_b\sin(\omega t - \gamma)], \qquad D_\mathrm{e} = R_\mathrm{r}(i_a^2 + i_b^2)/2,$$

where L_{r} and R_{r} are the inductance and resistance of the loops (windings) and i_b is the current in the phantom winding. Accordingly, the Lagrange–Maxwell equations for the independent coordinates take the form

$$L_{\mathrm{r}}\dot{i}_a = \Psi(\omega - \dot{\gamma})\sin(\omega t - \gamma) - R_{\mathrm{r}} i_a + u_f, \qquad L_{\mathrm{r}}\dot{i}_b = \Psi(\omega - \dot{\gamma})\cos(\omega t - \gamma) - R_{\mathrm{r}} i_b,$$
$$J\ddot{\gamma} = \Psi[i_a \sin(\omega t - \gamma) + i_b \cos(\omega t - \gamma)] - M_{\mathrm{load}}. \tag{10}$$

Similarly, as was done above, we write Eqs. (10) in dimensionless form

$$\dot{i}_a = -\alpha_{\mathrm{r}} i_a + bs\sin\theta + \alpha_{\mathrm{r}} u_f, \qquad \dot{i}_b = -\alpha_{\mathrm{r}} i_b + bs\cos\theta,$$
$$\dot{\theta} = s, \qquad \dot{s} = -\delta(i_a \sin\theta + i_b\cos\theta - M_{\mathrm{load}}), \qquad \alpha_{\mathrm{r}} = R_{\mathrm{r}}/(\omega L_{\mathrm{r}}). \tag{11}$$

The stability in small of system (11) in the case $u_f > 0$ is studied in the same manner as was done for system (6). In the steady-state regime, from (11) it follows that $s = 0$, $i_a = u_f$, $i_b = 0$, and $u_f \sin\theta_0 = M_{\mathrm{load}}$. The corresponding linearized equations are written as

$$\dot{\tilde{i}}_a = -\alpha_{\mathrm{r}}\tilde{i}_a + b\tilde{s}\sin\theta_0, \qquad \dot{\tilde{i}}_b = -\alpha_{\mathrm{r}}\tilde{i}_b + b\tilde{s}\sin\theta_0,$$
$$\dot{\tilde{\theta}} = \tilde{s}, \qquad \dot{\tilde{s}} = -\delta(\tilde{i}_a\sin\theta_0 + \tilde{i}_b\cos\theta_0 + u_f\tilde{\theta}\cos\theta_0).$$

From the characteristic equation of the above linear system

$$(\lambda + \alpha_{\mathrm{r}})(\lambda^3 + \alpha_{\mathrm{r}}\lambda^2 + \delta(b + u_f\cos\theta_0)\lambda + \delta u_f \alpha_{\mathrm{r}}\cos\theta_0) = 0$$

it follows that a necessary and sufficient stability condition in small of the steady-state solutions of the equations of synchronous motors is the condition $\cos\theta_0 > 0$, i.e., as in real synchronous motors, a small load cannot cause instability (only the condition $0 \leqslant M_{\mathrm{load}} < u_f$ is necessary).

Thus, we proved the need for introduction of a phantom winding in modeling the rotor of synchronous machines.

It should be noted that the well-known equations describing the motion of a rotor, as a rule, contain a synchronizing moment in the form of $u_f \sin\theta$. To explicitly distinguish this synchronizing moment in system (11), too, we make the change of variables $i_a = i'_a + u_f$ in (11). Omitting primes, we obtain

$$\dot{i}_a = -\alpha_{\mathrm{r}} i_a + bs\sin\theta, \qquad \dot{i}_b = -\alpha_{\mathrm{r}} i_b + bs\cos\theta,$$
$$\dot{\theta} = s, \qquad \dot{s} = -\delta(i_a\sin\theta + i_b\cos\theta + u_f\sin\theta - M_{\mathrm{load}}).$$

This transformation implies translation of the force generalized over the coordinate i_a to the coordinate s. As a result, the generalized force acquires the meaning of an additional synchronizing moment on the rotor shaft. In this case, for no-load operation ($M_{\mathrm{load}} = 0$) in a steady-state mode, the new value of the current is $i_a = 0$, as is the case in an induction motor.

We consider the case of an induction motor ($u_f = 0$). Let us reduce system (11) to a more convenient a form for further analysis. Introducing the variables x and y by the formulas

$$(-\alpha_{\mathrm{r}} x + 1)b = -i_a\cos\theta + i_b\sin\theta, \qquad \alpha_{\mathrm{r}} b y = i_a\sin\theta + i_b\cos\theta,$$

we obtain

$$\dot{x} = -\alpha x - sy + 1, \quad \dot{y} = -\alpha y + sx, \quad \dot{\theta} = s, \quad s = -\delta(\alpha b y - M_{\mathrm{load}}). \tag{12}$$

For nonlocal analysis of the mathematical model (12), we write it as a third-order system that does not contain the variable θ:

$$\dot{x} = -\alpha x - sy + 1, \quad \dot{y} = -\alpha y + sx, \quad \dot{s} = -\delta(\alpha b y - M_{\mathrm{load}}). \tag{13}$$

Here x and y are quasicurrents in the stator windings, s is the rotor slide, $\alpha = \alpha_{\mathrm{r}}$ is the resistance of the rotor windings, δ is the electromechanical constant that is inversely proportional to the moment of inertia of the rotor, and M_{load} is the dimensionless external load moment on the rotor shaft.

Under the assumptions made, stability analysis of the complete system of equations of an induction motor can be reduced to stability analysis of the system of third-order equations (13).

Let us consider the complete mathematical model of an induction motor as a system of sixth-order nonlinear ordinary differential equations that describes the dynamics of an induction motor in conventional idealized concepts, which are described in detail in, e.g., [3, pp. 28–36; 5, pp. 142–156; 6]. The main of them are as follows: 1) assumption of the invariance of an electromagnetic field in any cross section of the idealized physical model of an induction motor ignoring edge effects (hypothesis of a flat model); 2) assumption on the possibility of describing the interplay of electromagnetic processes in the windings of the machine stator and rotor using two symmetric, linear electric circuits.

Let us consider the following system of differential equations:

$$
\begin{gathered}
\Big(L_{\mathrm{s}}\frac{d}{dt}+R_{\mathrm{s}}\Big)i_\alpha^{\mathrm{s}}+æM\frac{d}{dt}\,(i_\alpha^{\mathrm{r}}\cos\gamma-i_\beta^{\mathrm{r}}\sin\gamma)=-u_m\sin{(\omega t)},\\
\Big(L_{\mathrm{s}}\frac{d}{dt}+R_{\mathrm{s}}\Big)i_\beta^{\mathrm{s}}+æM\frac{d}{dt}\,(i_\alpha^{\mathrm{r}}\sin\gamma+i_\beta^{\mathrm{r}}\cos\gamma)=u_m\cos{(\omega t)},\\
M\frac{d}{dt}\,(i_\alpha^{\mathrm{s}}\cos\gamma+i_\beta^{\mathrm{s}}\sin\gamma)+\Big(L_{\mathrm{r}}\frac{d}{dt}+R_{\mathrm{r}}\Big)i_\alpha^{\mathrm{r}}=0,\\
M\frac{d}{dt}(-i_\alpha^{\mathrm{s}}\sin\gamma+i_\beta^{\mathrm{s}}\cos\gamma)+\Big(L_{\mathrm{r}}\frac{d}{dt}+R_{\mathrm{r}}\Big)i_\beta^{\mathrm{r}}=0,\\
J\ddot{\gamma}=M[(i_\alpha^{\mathrm{r}}i_\beta^{\mathrm{s}}-i_\beta^{\mathrm{r}}i_\alpha^{\mathrm{s}})\cos\gamma-(i_\alpha^{\mathrm{r}}i_\alpha^{\mathrm{s}}+i_\beta^{\mathrm{r}}i_\beta^{\mathrm{s}})\sin\gamma]-M_{\mathrm{load}}.
\end{gathered}
\tag{14}
$$

Here i_α^{s}, i_β^{s}, i_α^{r}, and i_β^{r} are the stator and rotor winding currents, γ is the angle of rotation of the rotor, R_{s}, L_{s}, R_{r}, and L_{r} are the resistances and inductances of the corresponding windings, M is the amplitude of the mutual inductance, J is the moment of inertia of the induction motor rotor, $\omega = 2\pi f$, f and u_m are the frequency and amplitude of the voltage applied to the stator windings, respectively, M_{load} is the load moment on the induction motor shaft, t is current time, and æ is a parameter that characterizes the effect of electromagnetic processes in the rotor on the processes in the stator windings. Equations (14) coincide with Eqs. (8-1c), (8-2d) in [6] with accuracy up to designations.

For further transformations of Eqs. (14), we need expressions for the interlinkage of the windings. In the case of a cylindrical rotor EM (for example, in the case of an induction motor), these interlinkages can be obtained from formulas (3-3a)–(3-3d) in [6] taking into account the expressions for the inductances (3-40)–(3-49) [6]. In the above designation, they are written as

$$
\begin{aligned}
\psi_\alpha^{\mathrm{s}}&=L_{\mathrm{s}}i_\alpha^{\mathrm{s}}+æM(i_\alpha^{\mathrm{r}}\cos\gamma-i_\beta^{\mathrm{r}}\sin\gamma), &\quad \psi_\beta^{\mathrm{s}}&=L_{\mathrm{s}}i_\beta^{\mathrm{s}}+æM(i_\alpha^{\mathrm{r}}\sin\gamma+i_\beta^{\mathrm{r}}\cos\gamma),\\
\psi_\alpha^{\mathrm{r}}&=M(i_\alpha^{\mathrm{s}}\cos\gamma+i_\beta^{\mathrm{s}}\sin\gamma)+L_{\mathrm{r}}i_\alpha^{\mathrm{r}}, &\quad \psi_\beta^{\mathrm{r}}&=M(-i_\alpha^{\mathrm{s}}\sin\gamma+i_\beta^{\mathrm{s}}\cos\gamma)+L_{\mathrm{r}}i_\beta^{\mathrm{r}}.
\end{aligned}
\tag{15}
$$

Equations (14) and the expressions for the interlinkages (15) are written in so-called phase coordinates α, β. They are inconvenient for mathematical analysis but, in our case, can be simplified by Park's nonholonomic transformation of coordinates [3]. In the case of an induction motor, this can be done by introducing auxiliary orthogonal axes u and v, which rotate with an arbitrary angular velocity and make angle γ_k with the magnetic axis of the phase α of the induction motor stator. Such transformation is given, for example, in [6]. In the designations used in the present paper, it takes the form

$$
\begin{aligned}
i_u^{\mathrm{s}}&=i_\alpha^{\mathrm{s}}\cos\gamma_k+i_\beta^{\mathrm{s}}\sin\gamma_k, &\quad i_v^{\mathrm{s}}&=-i_\alpha^{\mathrm{s}}\sin\gamma_k+i_\beta^{\mathrm{s}}\cos\gamma_k,\\
i_u^{\mathrm{r}}&=i_\alpha^{\mathrm{r}}\cos{(\gamma_k-\gamma)}+i_\beta^{\mathrm{r}}\sin{(\gamma_k-\gamma)}, &\quad i_v^{\mathrm{r}}&=-i_\alpha^{\mathrm{r}}\sin{(\gamma_k-\gamma)}+i_\beta^{\mathrm{r}}\cos{(\gamma_k-\gamma)},\\
\psi_u^{\mathrm{s}}&=\psi_\alpha^{\mathrm{s}}\cos\gamma_k+\psi_\beta^{\mathrm{s}}\sin\gamma_k, &\quad \psi_v^{\mathrm{s}}&=-\psi_\alpha^{\mathrm{s}}\sin\gamma_k+\psi_\beta^{\mathrm{s}}\cos\gamma_k,\\
\psi_u^{\mathrm{r}}&=\psi_\alpha^{\mathrm{r}}\cos{(\gamma_k-\gamma)}+\psi_\beta^{\mathrm{r}}\sin{(\gamma_k-\gamma)}, &\quad \psi_v^{\mathrm{r}}&=-\psi_\alpha^{\mathrm{r}}\sin{(\gamma_k-\gamma)}+\psi_\beta^{\mathrm{r}}\cos{(\gamma_k-\gamma)},
\end{aligned}
$$

$$
u_u=u_m(-\sin{(\omega t)}\cos\gamma_k+\cos{(\omega t)}\sin\gamma_k)=-u_m\sin{(\omega t-\gamma_k)},
$$

$$
u_v=u_m(\sin{(\omega t)}\sin\gamma_k+\cos{(\omega t)}\cos\gamma_k)=u_m\cos{(\omega t-\gamma_k)}.
$$

This transformation implies that the real (phase) variables written in two orthogonal coordinate systems, one of which is rigidly fastened to the induction motor stator, and the other to its rotor, are converted to the projections of these variables (quasivariables) onto the same orthogonal axes u and v. In these coordinates (quasicurrents and quaisinterlinkages), Eq. (14) and expressions (15) become

$$\begin{gathered} L_{\rm s}(\dot{i}_u^{\rm s} - \dot{\gamma}_k i_v^{\rm s}) + R_{\rm s} i_u^{\rm s} æ M(\dot{i}_u^{\rm r} - \dot{\gamma}_k i_v^{\rm r}) = -u_m \sin(\omega t - \gamma_k), \\ L_{\rm s}(\dot{i}_v^{\rm s} + \dot{\gamma}_k i_u^{\rm s}) + R_{\rm s} i_v^{\rm s} + æ M(\dot{i}_v^{\rm r} + \dot{\gamma}_k i_u^{\rm r}) = u_m \cos(\omega t - \gamma_k), \\ M[\dot{i}_u^{\rm s} - (\dot{\gamma}_k - \dot{\gamma}) i_v^{\rm s}] + R_{\rm r} i_u^{\rm r} + L_{\rm r}[\dot{i}_u^{\rm r} - (\dot{\gamma}_k - \dot{\gamma}) i_v^{\rm r}] = 0, \\ M[\dot{i}_v^{\rm s} + (\dot{\gamma}_k - \dot{\gamma}) i_u^{\rm s}] + R_{\rm r} i_v^{\rm r} + L_{\rm r}[\dot{i}_v^{\rm r} + (\dot{\gamma}_k - \dot{\gamma}) i_u^{\rm r}] = 0, \\ J\ddot{\gamma} = M(i_u^{\rm r} i_v^{\rm s} - i_v^{\rm r} i_u^{\rm s}) - M_{\rm load}; \end{gathered} \tag{16}$$

$$\psi_u^{\rm s} = L_{\rm s} i_u^{\rm s} + æ M i_u^{\rm r}, \quad \psi_v^{\rm s} = L_{\rm s} i_v^{\rm s} + æ M i_v^{\rm r}, \quad \psi_u^{\rm r} = M i_u^{\rm s} + L_{\rm r} i_u^{\rm r}, \quad \psi_v^{\rm r} = M i_v^{\rm s} + L_{\rm r} i_v^{\rm r}. \tag{17}$$

In Eqs. (16) and formulas (17), we convert to dimensionless variable using the formulas

$$\tau = \omega t, \quad \frac{d\gamma}{d\tau} = 1 - s, \quad \psi^{\rm s} = \frac{u_m}{\omega} \bar{\psi}_{\rm s}, \quad \psi^{\rm r} = \frac{u_m}{\omega} \frac{L_{\rm r}}{M} \bar{\psi}^{\rm r}, \quad i^{\rm s} = \frac{u_m}{\omega L_{\rm s}} \bar{i}^{\rm s}, \quad i^{\rm r} = \frac{u_m}{\omega} \frac{1}{M} \bar{i}^{\, r},$$

$$\varepsilon_{\rm s} = \frac{R_{\rm s}}{\omega L_{\rm s}}, \quad \varepsilon_{\rm r} = \frac{R_{\rm r}}{\omega L_{\rm r}}, \quad \delta = \frac{u_m^2}{\omega^4 J L_{\rm s}}, \quad \bar{M}_{\rm load} = \frac{M_{\rm load}}{\omega^2 J \delta}, \quad \mu = 1 - \frac{M^2}{L_{\rm s} L_{\rm r}}$$

(μ is the coefficient of electromagnetic scattering in the air gap of the induction motor).

Omitting the bar above dimensionless quantities, we write Eqs. (16) and relation (17) in the form

$$\begin{gathered} \dot{i}_u^{\rm s} - \dot{\gamma}_k i_v^{\rm s} + \varepsilon_{\rm s} i_u^{\rm s} + æ(\dot{i}_u^{\rm r} - \dot{\gamma}_k i_v^{\rm r}) = -\sin(\tau - \gamma_k), \\ \dot{i}_v^{\rm s} + \dot{\gamma}_k i_u^{\rm s} + \varepsilon_{\rm s} i_v^{\rm s} + æ(\dot{i}_v^{\rm r} - \dot{\gamma}_k i_u^{\rm r}) = \cos(\tau - \gamma_k), \\ (1-\mu)[\dot{i}_u^{\rm s} - (\dot{\gamma}_k - 1 + s) i_v^{\rm s}] + \varepsilon_{\rm r} i_u^{\rm r} + \dot{i}_u^{\rm r} - (\dot{\gamma}_k - 1 + s) i_v^{\rm r} = 0, \\ (1-\mu)[\dot{i}_v^{\rm s} + (\dot{\gamma}_k - 1 + s) i_u^{\rm s}] + \varepsilon_{\rm r} i_v^{\rm r} + \dot{i}_v^{\rm r} + (\dot{\gamma}_k - 1 + s) i_u^{\rm r} = 0, \\ \dot{s} = -\delta[(i_u^{\rm r} i_v^{\rm s} - i_v^{\rm r} i_u^{\rm s}) - \bar{M}_{\rm load}]; \end{gathered} \tag{18}$$

$$\psi_u^{\rm s} = i_u^{\rm s} + æ i_u^{\rm r}, \quad \psi_v^{\rm s} = i_v^{\rm s} + æ i_v^{\rm r}, \quad \psi_u^{\rm r} = (1-\mu) i_u^{\rm s} + i_u^{\rm r}, \quad \psi_v^{\rm s} = (1-\mu) i_v^{\rm s} + i_v^{\rm r}, \tag{19}$$

where $\psi_u = \psi_u(\tau)$, $\psi_v = \psi_v(\tau)$, $i_u = i_u(\tau)$, and $i_v = i_v(\tau)$ are the quasiinterlinkages and quasicurrents of the corresponding windings, $s(\tau)$ is the slide (relative difference of the angular velocities of the rotor and the magnetic field of the stator), τ is the dimensionless time (angle of rotation of the stator magnetic field), $\varepsilon_{\rm s}$ and $\varepsilon_{\rm r}$ are the resistances of the stator and rotor windings, and γ_k is the angle of rotation of the axes u and v.

In (18), we set $\gamma_k = \tau$, i.e., we convert to so-called synchronous coordinate axes x and y (rotating synchronously with the stator magnetic field):

$$\begin{gathered} \dot{i}_x^{\rm s} - i_y^{\rm s} + \varepsilon_{\rm s} i_x^{\rm s} + æ(\dot{i}_x^{\rm r} - i_y^{\rm r}) = 0, \quad \dot{i}_y^{\rm s} + i_x^{\rm s} + \varepsilon_{\rm s} i_y^{\rm s} + æ(\dot{i}_y^{\rm r} + i_x^{\rm r}) = 1, \\ (1-\mu)(\dot{i}_x^{\rm s} - s i_y^{\rm s}) + \varepsilon_{\rm r} i_x^{\rm r} + \dot{i}_x^{\rm r} - s i_y^{\rm r} = 0, \quad (1-\mu)(\dot{i}_y^{\rm s} + s i_x^{\rm s}) + \varepsilon_{\rm r} i_y^{\rm r} + \dot{i}_y^{\rm r} + s i_x^{\rm r} = 0, \\ \dot{s} = -\delta[(i_x^{\rm r} i_y^{\rm s} - i_y^{\rm r} i_x^{\rm s}) - \bar{M}_{\rm load}]; \end{gathered} \tag{20}$$

$$\psi_x^{\rm s} = i_x^{\rm s} + æ i_x^{\rm r}, \quad \psi_y^{\rm s} = i_y^{\rm s} + æ i_y^{\rm r}, \quad \psi_x^{\rm r} = (1-\mu) i_x^{\rm s} + i_x^{\rm r}, \quad \psi_y^{\rm r} = (1-\mu) i_y^{\rm s} + i_y^{\rm r}. \tag{21}$$

To examine Eqs. (20), it is more convenient to convert to the quasicurrents of the stator windings and quasiinterlinkages of the rotor windings, by eliminating the quasicurrents of the rotor windings using expressions (21):

$$i_x^{\rm r} = \psi_x^{\rm r} - (1-\mu) i_x^{\rm s}, \qquad i_y^{\rm r} = \psi_y^{\rm r} - (1-\mu) i_y^{\rm s}.$$

As a result, we obtain the equations of an induction motor in so-called hybrid variables, in which the quasicurrents $i_x^{\rm s}$ and $i_y^{\rm s}$ are replaced by the variables x and y given by the formulas

$$x = \mu' i_x^{\rm s} + æ \psi_x^{\rm r}, \quad y = \mu' i_y^{\rm s} + æ \psi_y^{\rm r}, \quad \mu' = 1 - æ(1-\mu).$$

Then, Eqs. (20) become

$$\dot{x} = -\alpha'_{\rm s}x + y + æ\alpha'_{\rm s}\psi^{\rm r}_x, \qquad \dot{y} = -\alpha'_{\rm s}y - x + æ\alpha'_{\rm s}\psi^{\rm r}_y + 1,$$

$$\dot{\psi}^{\rm r}_x = -\alpha'_{\rm r}\psi^{\rm r}_x + s\psi^{\rm r}_y + \alpha'_{\rm r}(1-\mu)x, \quad \dot{\psi}^{\rm r}_y = -\alpha'_{\rm r}\psi^{\rm r}_y - s\psi^{\rm r}_x + \alpha'_{\rm r}(1-\mu)y, \tag{22}$$

$$\dot{s} = -\delta[(\psi^{\rm r}_x y - \psi^{\rm r}_y x)/\mu' - \bar{M}_{\rm load}],$$

where $\alpha'_{\rm s} = \varepsilon_{\rm s}/\mu'$ and $\alpha'_{\rm r} = \varepsilon_{\rm r}/\mu'$.

Using the change of variables

$$\bar{x} = \frac{\psi^{\rm r}_x + \alpha'_{\rm s}\psi^{\rm r}_y}{\alpha'_{\rm r}(1-\mu)}, \quad \bar{y} = \frac{\psi^{\rm r}_y - \alpha'_{\rm s}\psi^{\rm r}_x}{\alpha'_{\rm r}(1-\mu)}, \quad \psi_x = \frac{x + \alpha'_{\rm s}y - 1}{\alpha'_{\rm r}(1-\mu)}, \quad \psi_y = \frac{y - \alpha'_{\rm s}x}{\alpha'_{\rm r}(1-\mu)},$$

we reduce system (22) to the more convenient form

$$\dot{\psi}_x = -\alpha_{\rm s}\psi_x + \psi_y + æ\alpha_{\rm s}x, \qquad \dot{\psi}_y = -\alpha_{\rm s}\psi_y - \psi_x + æ\alpha_{\rm s}y,$$

$$\dot{x} = -\alpha_{\rm r}x + sy + \alpha_{\rm r}(1-\mu)\psi_x + 1, \qquad \dot{y} = -\alpha_{\rm r}y - sx + \alpha_{\rm r}(1-\mu)\psi_y, \tag{23}$$

$$\dot{s} = \delta[\alpha_{\rm r}by + \alpha_{\rm r}^2 b(1-\mu)(\psi_x y - \psi_y x) + M_{\rm load}].$$

Here $x = \bar{x}$, $y = \bar{y}$, $\alpha_{\rm s} = \alpha'_{\rm s}$, $\alpha_{\rm r} = \alpha'_{\rm r}$, $M_{\rm load} = \bar{M}_{\rm load}$, and $b = (1-\mu)/[\mu'(1+\alpha'^2_{\rm s})]$.

Along with system (23), we consider the simpler system

$$\dot{\psi}_x = -\alpha_{\rm s}\psi_x + \psi_y, \qquad \dot{\psi}_y = -\alpha_{\rm s}\psi_y - \psi_x; \tag{24}$$

$$\dot{x} = -\alpha_{\rm r}x + sy + \alpha_{\rm r}(1-\mu)\psi_x + 1, \qquad \dot{y} = -\alpha_{\rm r}y - sx + \alpha_{\rm r}(1-\mu)\psi_y,$$

$$\dot{s} = \delta[\alpha_{\rm r}by + \alpha_{\rm r}^2 b(1-\mu)(\psi_x y - \psi_y x) + M_{\rm load}]. \tag{25}$$

System (24), (25) is the zero approximation ($æ = 0$) of system (23) in the small regular parameter $æ$. The assumption $æ = 0$, in essence, implies the absence of the effect of electromagnetic processes in the rotor on the processes in the stator [see the first two equations in (14)], which is used in modeling the stator by a rotating constant magnet.

Let us consider a simplified mathematical model that describes the slow (compared with electrical) mechanical motion of the rotor. Apparently, Eqs. (24) for the quasiinterlinkage of the stator windings of an induction motor are globally stable and have the steady-state solution $\psi^0_x = 0$, $\psi^0_y = 0$. Therefore, following the conventional quasisteady-state approach to studying the complete split equations (24) and (25), one often substitutes the steady-state solution of system (24) into Eqs. (25).

Thus, for an induction motor, we obtain system (13), in which $\alpha = \alpha_{\rm r}$.

The smallness of the parameter δ (inversely proportional to the moment of inertia of the rotor) underlies the above-mentioned engineering approach to simplifying system (13) in studying the oscillations of the induction motor rotor and its stability. This approach consists of freezing the slow mechanical variable s in the equations for the rapid electrical variables x and y and using the value of y for steady-state conditions in the last of Eqs. (13):

$$\dot{s} = -\delta(\alpha bs/(\alpha^2 + s^2) - M_{\rm load}). \tag{26}$$

For the steady-stated regime, from (26) it follows that

$$M_{\rm load} = \alpha bs/(\alpha^2 + s^2). \tag{27}$$

The parameter $M_{\rm load}$ is a conventional static mechanical characteristics of an induction motor. From (27) it follows that the electromagnetic moment reaches the maximum value $M^{\rm max}_{\rm load}$ at $s = s_{\rm cr} = \alpha$. According to the Government Standard, the parameter s is denoted by $s_{\rm cr}$ and is given in the log of each induction motor.

The relation

$$m = M_{\rm load}/M^{\rm max}_{\rm load} = 2ss_{\rm cr}/(s^2_{\rm cr} + s^2) = 2/(s_{\rm cr}/s + s/s_{\rm cr}),$$

known in the theory of an induction electric motor is called Kloss's formula.

In formulating results, a system will be called dichotomic if any of its solutions bounded on the positive semiaxis tends to a steady-state set, and a system is called globally asymptotically stable if any solution of this system tends to a certain equilibrium state.

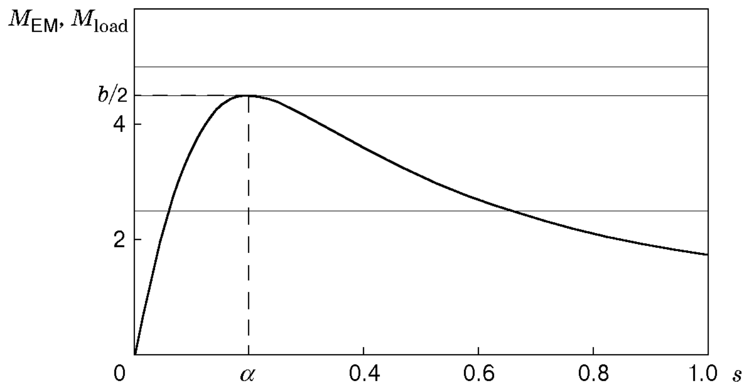

Fig. 3

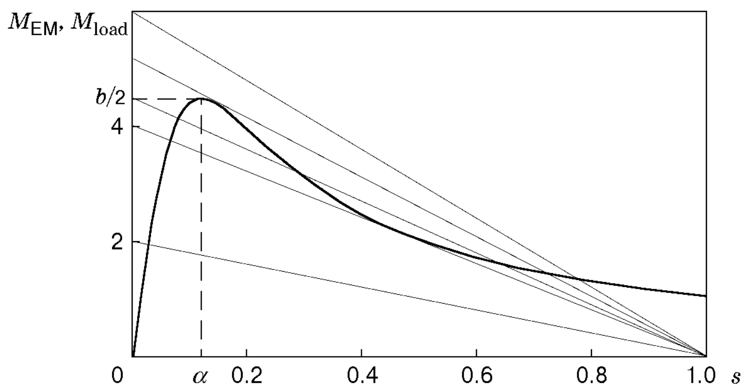

Fig. 4

Let us consider Eq. (26).

Theorem 1. *Let* $M_{\text{load}} = \text{const}$. *Then:* 1) *Eq.* (26) *is generally stable if* $M_{\text{load}} = 0$; 2) *Eq.* (26) *is dichotomic if* $0 < M_{\text{load}} < b/2$; 3) *Eq.* (26) *has a single semistable equilibrium state if* $M_{\text{load}} = b/2$; 4) *Eq.* (26) *does not have of bounded solutions if* $M_{\text{load}} > b/2$.

Proof. 1. Let $M_{\text{load}} = 0$. The point $s = 0$ is a single steady-state point of Eq. (26). The function $v = s^2/2$ has a negative derivative along any trajectory. Therefore, the equation is generally stable.

2. Let $0 < M_{\text{load}} < b/2$. The equilibrium states are $s = s_1$ (steady-state) and $s = s_2$ (unsteady). From the expression $v_i = (s - s_i)^2$ $(i = 1, 2)$ it follows that the interval $(-\infty, s_2)$ is the region of attraction of the equilibrium state $s = s_1$ and any trajectory $s(t, s_0)$, where $s_0 > s_2$, goes to infinity. Figure 3 shows the static mechanical characteristics of an induction motor (horizontal straight lines are constant load moments). It is evident that two, one or none points of intersection can exist.

3. Let $M_{\text{load}} = b/2$. Equation (26) has a single steady-state point $s = \alpha_{\text{r}}$, whose region of attraction is the interval $(-\infty, \alpha_{\text{r}})$. Any trajectory $s(t, s_0)$, where $s_0 > \alpha_{\text{r}}$ goes to infinity (Fig. 3). [Lyapunov's function $v = (s-\alpha_{\text{r}})^2$ is used.]

4. Let $M_{\text{load}} > b/2$. Equation (26) has no steady-state points. The derivative of the function $v = s^2$ is positive along any trajectory.

Theorem 2. *Let* $M_{\text{load}} = k(1 - s)$, *where* $k = \text{const}$ $(k > 0)$. *Then:* 1) *Eq.* (26) *is generally stable if one steady-state point exists;* 2) *Eq.* (26) *is dichotomic if two steady-state points exist;* 3) *Eq.* (26) *is globally asymptotically stable if three steady-state points exist* (Fig. 4).

The proof of Theorem 2 is similar to the proof of Theorem 1 with the use of Lyapunov's function of the same form.

In case 1, the region of attraction of the point s_0 is the entire phase axis $(-\infty, +\infty)$. In case 2, the region of attraction of the stable equilibrium state $s = s_1$ is the interval $(-\infty, s_2)$. The equilibrium state $s = s_2$ is semistable. In case 3, the steady-state points s_1 and s_3 are stable and have the regions of attraction $(-\infty, s_2)$ and $(s_2, +\infty)$, respectively. The steady-state point s_2 is unstable.

Let us consider system (13). We note that the structure of system (13) is similar to that of the well-known Lorentz system [7]. This allows us to perform a nonlocal analysis of this system using Lyapunov's functions considered in [8] and to obtain the conditions of dichotomy and global asymptotic stability.

The third-order system (13) for the cases of constant external load moment and the load moment dependent linearly on slide is studied in [9–11]. In [10, 11], it is shown that stability analysis of the fifth-order system (24) (25) reduces to stability analysis of the third-order system (13), i.e., under the same conditions, the same statements are valid for system (13) and system (24), (25). For convenience, we give theorems that formulate the indicated results for systems (13) and (24), (25).

Theorem 3. *Let* $M_{\text{load}} = \text{const}$. *Then, the following statements hold:* 1) *if* $0 < M_{\text{load}} < \min\{b/2, 2\alpha^2/\delta\}$, *systems* (13) *and* (24), (25) *are dichotomic;* 2) *if* $b/2 < M_{\text{load}} < 2\alpha^2/\delta$, *all solutions of systems* (13) *and* (24), (34) *are unbounded.*

Theorem 4. *Let* $M_{\text{load}} = \bar{k}(1-s)$, $\bar{k} = \text{const}$. *Then, if* $0 < \bar{k}\delta \leqslant 3\alpha\sqrt{(\alpha + \bar{k}\delta)(\alpha - 2\bar{k}\delta)}$, *systems* (13) *and* (24), (25) *are globally asymptotically stable.*

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